Penelope Maddy, *Defending the Axioms: On the Philosophical Foundations of Set Theory*, Oxford University Press, 2011, pp. 150, $ 45.00, ISBN 9780199596188

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This little but dense book from philosopher of mathematics Penelope Maddy proposes an original stance about ontology and epistemology of mathematics, but, in my opinion, it succeeds only partially, as will become clearer below. Nevertheless, this is a very interesting book, with some really good philosophical insight and very well articulated arguments, spanning over a wide range of problems in philosophy of mathematics. And some of the many digressions Maddy makes (which I have no space here to describe in detail) are enlightening.
The book is structured in five chapters that gradually lead to the deployment of a series of theses related to a more general position, already defined by the author (see Maddy 2007) "Second Philosophy" to contrast it with Metaphysics, the traditional First Philosophy in search of first principles. The first chapter of the book provides an account of the history of mathematics that highlights a pattern of multiple inversions occurred between applied and abstract mathematics from ancient Greeks to contemporary years: in the beginning, with Plato, mathematics was seen as a type of knowledge provided with more certainty than knowledge about concrete objects, thus, in that age, physics was, in a sense, a discipline subordinate to mathematics. In the Modern Age, beginning with Galileo, Descartes and Newton, it is physics which starts to require new mathematical developments, which are tailor-made to suit the needs of natural science: mathematics is at the time nothing more than applied mathematics, closely related to physics. This type of mathematical naturalism becomes problematic once, beginning with the XIX century, pure mathematics rises, for such a flavour of mathematics (non-euclidean geometry, the set theory of Cantor and Dedekind) cannot be viewed as a form of "physics" anymore. So, we move from the "mathematism" (the term is mine) of Plato, where certainty belongs to mathematical knowledge only, to modern mathematical naturalism, where mathematics is a by-product of physics, and certainty is the certainty of our best form of empirical knowledge—experimental science—to which mathematical certainty is now

seen as subordinate. So, when Mathematics frees itself from Physics to become abstract, a need to justify its findings arises, and with it the classical dilemma highlighted in a well-known paper by Benacerraf (Benacerraf 1973) regarding our epistemic access to mathematical truths and objects: if abstract mathematics refers to a world of abstract entities, with no causal connection with the empirical one, how can we come to know anything about this abstract world, given that human knowledge is based on a causal empirical process (at least according to science)? Historically, another reversal has occurred in contemporary science: mathematics, providing *theoretical models* with a limited applicability for empirical sciences, once again becomes part of the empirical research endeavor, but (as models are not *true* but only approximate and useful) it remains at the same time *abstract* mathematics, and as such still in need of justification. This need was of course the driving force behind the search for rigor and the answer to it: the rise of the axiomatic method and of Set Theory, seen as the foundation of all mathematics. But set theory itself has always been involved in controversies about the justification of some of its axioms, so the task is now to find arguments to defend set theory axioms themselves: how must introduction of new axioms be justified, and, once the right method is found, what philosophical justification of its validity must be provided?

Chapter II introduces the notion of *Second Philosophy* (*SP* from now on): a philosophy which, contrary to the tradition of First Philosophy, comes neither before nor after science to establish, independently, the founding principles or methods of scientific research, but that, even when it aims to correct or justify science, sees this task as just a part of a general enterprise to pursue knowledge, which starts with the image of an ideal subject that sets out to discover the world, beginning with ordinary belief and perception, gradually developing more sophisticated methods, and later starting to investigate that same methods and their validity: a type of naturalistic framework in which (second) philosophy comes after the scientific endeavor has developed its methods, as an enquiry about them. Inside such a form of naturalism, we still have to answer the Benacerraf's dilemma, i.e.: what are sets, that we can come to know about them through mathematical research? This requires resolving non-trivial epistemological and ontological problems. Maddy proposes two ontological positions regarding

mathematics that will later reveal to be methodologically equivalent: *Thin Realism* (chap. III) and *Arealism* (chap. IV).
For Thin Realism (let's call it *TR*) the answer to the above question is straightforward: “sets just are the sort of thing set theory describes; this is all there is to them.” (p.61). In contrast with traditional Platonic realism (a flavour of realism Maddy calls *Robust Realism, RR* for us here), TR avoids any other characterization of these abstract entities besides those positively asserted by set theory. RR, seeing the endeavor of mathematics as the discovery of some pre-existing world, “requires a non-trivial account of the reliability of set-theoretic methods, an account that goes beyond what set theory tells us; for the Thin Realist, set theory itself gives the whole story; the reliability of its methods is a plain fact about what sets are” (p.63). This way, TR avoids the need for a complex epistemological answer to Benacerraf's-type questions and remains firmly inside a naturalistic methodological view, refusing justifications external to the naturalistic frame. Maddy argues against the objection that TR can be viewed as “idealistic” (that sets are constituted by the set-theoretic practices themselves), or as a Carnap-like conventionalist position. Nevertheless, to dispel the residual doubt that sets are something not fully real, but a sort of shadow play thrown up by our set-theoretic methods, a question remains to be answered: what are sets, that the methods of set theory can track them? Maddy thinks TR can get “a sense of an objective reality underlying both the methods and the sets that illuminates the intimate connection between them” (p.77), which is the same as “asking what objective reality underlies and constrains set-theoretic methods” (*ibid.*): the answer for Maddy lies in a perhaps vague concept that she calls in various ways, but mainly *mathematical depth* (p.80). Perhaps metaphorically, she claims that set-theoretic methods allow us to track the “topography” (*ibid.*) or the “underlying contours” (p.82) of mathematical depth, and that this topography is “entirely objective” (p.80). This not further specified “objective reality” underlying mathematics, reveals itself through the objective constraints that guide set theory development towards the achievement of “greater mathematical depth, mathematical fruitfulness, mathematical effectiveness, mathematical importance, mathematical productivity, and so on” (p.81).
In chapter IV, Arealism (*AR*) is introduced: the position of a

Second Philosopher which would neither believe in the existence of sets nor in the truth of set-theoretic claims, because the methods of empirical research, the best methods for confirming existence and truth according to SP, cannot be applied to mathematics. Maddy carefully rejects the possibility that AR is a form of nominalism or a form of fictionalism, or a sort of what she calls *if-thenism* (the view that mathematics is only a matter of logically deducing something from some premise). The chapter goes on to conclude that, apart from the obvious difference between TR and AR regarding the belief in the truth of mathematical statements, the two positions are methodologically equivalent and indistinguishable, and that an objective reality underlies this condition: "the topography of mathematical depth […]. For the Thin Realist, sets are the things that mark these contours; set-theoretic methods are designed to track them. For the Arealist, these same contours are what motivate and guide her elaboration of the theory of sets" (p.100). The choice between TR and AR should then depend, for a Second Philosopher, on the answer to the question: is pure mathematics part of the empirical enquiry or do its methods happen to depart completely from it? After a long and ramified argumentation, Maddy concludes that there is "no substantive fact" (p.112) that could make us lean towards one or the other of the two ontological positions, for the same objective reality supports both choices: we are free to take sides.

In chapter V, after a long collateral discussion about objectivity in mathematics and the merits of a type of Fregean RR, which concludes in favour of TR/AR, Maddy introduces a last argument supporting the view that, contrary to tradition, in defending the axioms we should privilege *extrinsic* rather that intrinsic justifications. Using historical examples she shows that *extrinsic* consideration of mathematical depth (productivity, fruitfulness, and so on) are at play even when appeal is made to *intrinsic* reasons, namely in the form of what this intrinsic support (usually the sought-after self-evidence of the axiom) is supposed to provide: it's supposed to bring forth more mathematical fruitfulness anyway. This is a surprising conclusion, which I think is worth delving into.

Let me add another comment here: according to Maddy "questions of ontology and truth are red herrings" (p.117). She calls this position, resulting from undecidability between TR and AR, *Post-metaphisical objectivism* (p.123). Maddy thinks she has explained away two ancient and respectable sets of questions, but, in proposing the idea of *mathematical depth* as something objective that guides mathematical research, I think she has put aside the question on what is the source of this objectivity: a question that would require us to re-enter the muddy waters of ontology and epistemology. In my opinion, a promising route to follow is to try and see if ontological conclusions of some sort can or cannot be drawn from the indubitable objectivity of this "something deep" (p.131: the same as *mathematical depth*). Given that, for the moment, as Maddy notes, there's no compelling reason to prefer TR to AR, I think, following Occam's Razor rule, that TR should be discarded simply on the ground of its useless ontological and semantical assumptions, in favour of the more austere AR. This choice would have to be revised if, following the route I just sketched above, reasons to admit some sort of *ontological reality* underlying mathematics should be encountered: should an ontological reality of some kind that manifests itself to us through mathematics as "mathematical depth" in mathematical practice, constraining and guiding mathematical development as an invisible boundary, be inferred from the objectivity of such mathematical depth. I am aware this could go beyond what SP declares admissible, but I don't think such questions can be easily dismissed: mathematical depth is just a metaphor, but for what? What is that objectively constrains mathematical research? For Maddy "This form of objectivity is, as you might say, post-metaphysical" (p.116). I'm not sure we can afford to skip all metaphysical questions about mathematics from now on. To be fair, on p.117 Maddy admits she has only barely sketched the question of mathematical depth. I hope she is busy trying to elaborate on that, for I think that is a promising route.

This is a book well worth a reading, calling for long lasting reflections about what it means to do mathematics and what it means to do philosophy, be this practice of meta-reflection a form of Second or First Philosophy.

Bibliography

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