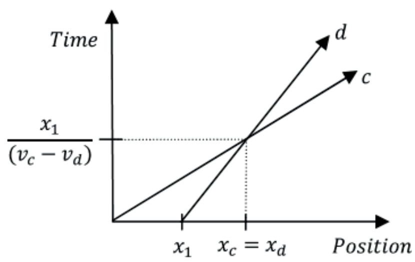
Vincenzo Fano, *I Paradossi di Zenone*, Carocci Editore, 2012, pp. 142, € 10.50, ISBN 9788843062676

Francesco Gallina, Università degli Studi di Padova

Vincenzo Fano's book, *I Paradossi di Zenone*, focuses on four paradoxes tracing back to Zeno, the Greek philosopher known for arguing against motion and plurality. The book reconstructs Zeno's arguments and compares them with several achievements in modern mathematics, classical physics and formal logic.

After a brief historical introduction (Ch.1), the author presents the so-called dichotomy paradox (Ch.2). Suppose that, at time t, particle c is moving with a finite, constant velocity v from point A toward point B. Assume that the distance x between A and B is finite. Will c ever reach point B? Of course it will! Everyday experience is all the evidence one needs to give a positive answer. However, Zeno argues, experience may be misleading. Indeed, for c to cover distance x, it needs to reach the midpoint, $x/2$. Similarly, to get to $x/2$, c is required to cover $x/4$. By reiterating this pattern, it follows that c has to go through an infinite number of finite space intervals during its motion from A toward B. Because c travels with a finite velocity, a finite amount of time will be needed to cover an infinite number of finite intervals of space. Therefore, particle c cannot reach B in a finite time interval. In fact, c cannot move at all, for each movement would require c to travel through an infinite number of space intervals in a finite time. Hence, the paradox. Fano notices that Zeno's argument lies on the false premise that the sum of an infinite number of finite (space) intervals is infinite. The previous example suggests that the sum of an infinite number of space intervals of the form $x - (x/2^n)$ tends to x when n approaches infinity. Therefore, c can cover a finite space interval x with a finite velocity v, reaching its destination B at $t' = v/x$. Although the solution to the dichotomy paradox might seem relatively simple, it reveals a non-trivial assumption about the nature of space. In order to guarantee that the sum of a series of (space) intervals is the limit to which its partial sums approach, one has to assume that space itself is continuous (p.28). By space continuity, the limit at which c's position tends as time approaches $t' = v/x$ is identical with c's position at t'.

In a similar way, Fano argues that mathematical analysis can be used to solve the Achilles paradox quite straightforwardly (Ch.3). At time t_1, particle c starts moving from point A toward point B, the distance between the two points being equal to a finite number x_1. Once again, c travels at constant and finite velocity v_c. Let d be another particle moving from B in the same trajectory as c at t_1. d has constant finite velocity v_d, with $v_d < v_c$. When at time $t_2 = x_1/v_c$ particle c reaches point B, d has covered the distance $x_2 = t_2 v_d = x_1(v_d / v_c)$. At a later moment $t_3 = x_2/v_c = x_1(v_d / v_c^2)$, c has covered the distance x_2, while d has covered the distance $x_3 = t_3 v_d = x_1(v_d^2 / v_c^2)$. This shows that the time interval needed for c to reach d equals the sum of an infinite number of finite time intervals. Since c moves with a finite velocity – Zeno concludes – it will never reach d. Analogously to the dichotomy paradox, the mistake resides in the mathematical assumption that the sum of an infinite number of finite (time) intervals is infinite. In order to get to d's position, c needs a time interval equals to the sum of intervals of the form $x_1(v_d^{n-1} / v_c^n)$, where n tends to infinity. However, such a sum equals to a finite number, $x_1/(v_c - v_d)$, and c will reach d precisely at the time corresponding to this value (p.48). One can figure out this scenario simply by using the diagram in the figure. The position of c at time t is $x_c = v_c t$, while d's position at t is $x_d = v_d t + x_1$. Once c reaches d, they both occupy the same space location. If $x_c = x_d$, then $v_c t = v_d t + x_1$. This equation yields to the time at which c reaches d, $t = x_1/(v_c - v_d)$. Similarly to the dichotomy paradox, the solution for the Achilles paradox is relatively simple. However, it presupposes a non-trivial assumption on the continuity of time (p.48).



The third paradox (Ch.4) had a relevant impact on the history of philosophy, mathematics and science. It has been subject of debate of several founding fathers of modern physics (such as Galileo Galilei), and it has been solved only thanks to the work of Georg Cantor. The paradox tries to demonstrate that a segment $\overline{AB}$ of non-null, finite length cannot be composed by

infinite zero-length points. If each point has zero length, the sum of infinite points would be zero. Is the argument valid? Modern analysis provides a negative answer. First, Cantor proved that there is a one-to-one map from the points in a straight line to the set of real numbers $\mathbb{R}$. Hence, one may associate segment $\overline{AB}$ with an interval (x, y) of real numbers. Interval (x, y) is the union of all its uncountable, many degenerate sub-intervals $\{z\}$ (where $x \leq z \leq y$). Its length is defined as the non-negative quantity $y - x$. Moreover, if an interval is the union of countably many disjoints intervals, its length is the limiting sum of the individual lengths of its sub-intervals. This latter property is called countable additivity, and it may raise the following (apparent) problem (p.85). By countable additivity, it seems impossible to compute the length of (x, y), since it would require to calculate the length of the union of *un*countable many (degenerate) sub-intervals. However, Cantor showed that any interval of reals is a union of at most countably many disjoint intervals. Therefore, there is no contradiction in saying that the length of an uncountable set may be finite. Cardinality and length are quite different magnitudes. In particular, the length of (x, y) (which corresponds to segment $\overline{AB}$) is $y - x$, which, in turn, may be equal to a finite quantity (p.87).
The last argument (Ch.5) is the only one among Zeno's paradoxes which has not yet being solved. It concerns motion and involves the following premises. According to the *incompatibility* principle, a particle c cannot be both in motion and at rest relative to the very same time t. The *determinacy* principle says that, at time t, c is either in motion or at rest. The *region* principle says that, if c at t is either in motion or at rest, there is a unique region of space s at which c is located at *t*. Finally, the *rest* principle affirms that, if there is a time t and a unique region of space s at which c is located, then c is at rest at that time. By *determinacy* and *region*, there is a unique region of space s at which c is located at t. But *rest* implies that c is at rest at t, and, via *incompatibility*, c is not moving at t. Movement is impossible. Clearly, Zeno's opponents have tried to refute (at least) one of the principles involved in the argument. However, Fano argues that it is not clear whether they have succeeded. In order to reject Zeno's conclusion, one possible move consists in refuting *rest* (p.107). Intuitively, if c has an instantaneous velocity at t, then c is moving at t (even if

it occupies a unique region of space at that time). Nevertheless, it is problematic whether the notion of instantaneous velocity, as it is defined in classical Newtonian physics, is useful within the dialectical context exemplified above. The instantaneous velocity v_i of a particle c at a time t is the limit of its average velocity, $\Delta x/\Delta t$, as Δt approaches 0. Δt must be different from 0, otherwise $\Delta x/\Delta t$ would be undefined. If Δt is different from 0, v_i is c's average velocity within a (however tiny) interval of time. Accordingly, v_i is not – strictly speaking – instantaneous. In other words, it is not entirely clear whether *rest* might be refuted via the current notion of instantaneous velocity (pp.108-109).

In conclusion, Fano's book is a very good introduction to Zeno's puzzles. It shows that any plausible account for motion and plurality requires developed mathematical techniques, along with a non-trivial understanding of physics. Nevertheless, some of the questions raised by Zeno have not been satisfactorily answered, and more work has to be done to avoid ancient worries.