Shay Allen Logan, *Relevance Logic*, Cambridge University Press, Cambridge 2024, pp. 86, € 64.19, ISBN 9781009536172

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Logan's *Relevance Logic* in the *Cambridge Elements* series offers a compelling introduction to the overcrowded world of relevance logic, one firmly rooted in a clear philosophical stance. The entire work is organized around a view that is unorthodox, at least historically: that the right way to understand relevance logics is not primarily through the lens of relevance, but through that of theory-building. Relevance, in particular, is treated as a byproduct of a more fundamental feature of the system – namely, its capacity to combine theories, meant as *fully fleshed out sets of sentences*, in appropriate ways. The book focuses on the weak relevant logic B as its central object of study, making it one of the more unusual introductory treatments in a field where R and E have historically dominated attention.

The work is organized in eight sections in a way that is almost polyphonic: the first two establish the general frame for the argumentative voices that unfold in the subsequent sections up to Section 7, while Section 8 offers a survey of alternative approaches not treated in comparable detail. This counterpoint also operates within the individual sections, where formal results and philosophical interpretation remain in constant interplay.

While the technical presentation is in a certain sense self-contained, it does presuppose some familiarity with standard proof-theoretic and model-theoretic techniques and results, which are introduced but not derived. This makes the book an excellent introductory text, especially for readers already familiar with the main metatheoretical results – at least at the propositional level. The extension to the first-order level, achieved through the treatment of quantifiers, is one of the book's notable merits, as such a treatment is rarely found elsewhere in the literature on relevance logic.

The first section opens with the observation that logic is commonly taken to rest on an imprecise yet intuitive notion: that valid inferences must be unimpeachable – that is, flawless and beyond challenge. The central concern of the book, however, is not directed at one of the founding issues of relevance logic as identified by its fathers, Anderson and Belnap – namely, the connection that must hold between antecedent and consequent – but at a broader question: the place that logic should occupy among our theoretical practices. In this spirit, one key observation is that unimpeachability is not guaranteed by relevance alone, nor is irrelevance sufficient to undermine it.

The second section is broader in scope, aiming to chart the philosophical landscape underlying the entire approach. It draws heavily on Jc Beall's conception of logic as a "universal basement-level closure relation" – the ur-relation on which all theory-building practices are grounded. The work then offers an informal characterization of the model-theoretic structures at play, understood as models of spaces of theories.

The third section exemplifies the approach described above, in which informal explanations are made operative within a formal context. Where many textbooks would begin by establishing the syntactic status of logic – a framework in which axiomatic presentation helps navigate the landscape of inclusion relations between logics – this section instead introduces models of spaces of theories equipped with a binary application operation and a containment relation together with a distinguished theory that serves as a left identity for application. The intuitive presentation of the model is followed by some useful results that will be deployed in the proof of completeness.

The fourth section presents the syntactic framework of the logic and culminates in its principal metatheoretical result, the completeness theorem for B. A particularly valuable pedagogical feature is the extended treatment of **MaGIC** (Slaney's matrix generator), which Logan recommends as the main instrument for checking non-theoremhood. The more technical part of the section is devoted to the proof of completeness by means of a canonical model. Here Logan develops the argument through theory closure, formal and prime theories, and their role in the canonical construction, thereby offering a clear bridge between the syntactic and semantic sides of the system. The section closes with a brief discussion of several extensions of the basic logic.

The fifth section is technically the richest. Logan states the classical *Variable Sharing Property* for R, according to which, if $A \rightarrow B \in R$, then A and B share an atomic variable. He then proves this by establishing, along the lines of Belnap's original approach via the eight-element partially ordered set M_0, the stronger *Strong Variable Sharing* theorem: the shared variable must occur with the same sign in both A and B. The most original material concerns *Depth Substitution Invariance*, a result Logan attributes to work building on Brady, with his own contributions in Logan (2021, 2022). A depth substitution is one that varies depending on how deeply an atom is nested within conditionals – so the same atom can be replaced by different formulas in different occurrences within a single formula. The surprising result is that B and DW are closed under depth substitutions, while logics with Suffixing, Prefixing, Contraction, or Commutativity are not (adding any of these to closure under depth substitution, closure under R1, and instances of A → A yields triviality). Logan's philosophical interpretation of this result is interesting: it argues that the minimal theory-building theory cannot treat the same atom as meaning the same thing at different depths, because different theories may interpret it differently at each depth. This is motivated with a pleasant digression involving arithmetic under two different interpretations. The section closes with a proof that, in B and DW, the theory generated by an atom or its negation is prime – a result applied later – and a discussion of connections to constructivism, including the Disjunction Property and a partial analogue of Harrop's Rule.

The sixth section introduces first-order models through stratified semantics, following Fine (1988) and Logan (2019). The core observation motivating stratification is that whether a set of sentences counts as a theory depends on the available stock of constants – one cannot assess whether "everything is purple" belongs to a theory without knowing which constants exist. This forces a hierarchy of languages and models indexed by finite sets of constants, connected by extension (↑) and restriction (↓) functions, plus symmetrization functions that identify pairs of constants. The resulting quantifier clause is non-Tarskian: ∀xA is verified at a theory t in stratum X just in case extending to a new constant ω_i (for some $i \notin X$) gives a theory that verifies $A(x/\omega_i)$. Logan persuasively argues that this clause is the natural one from the theory-building perspective: containing a universal requires that any new name witnesses it. The machinery is substantial – 22 conditions on stratified models – but the presentation succeeds in making this complexity manageable.

The seventh section axiomatizes the first-order logic BQ and establishes its completeness with respect to stratified models. Logan wisely focuses the completeness proof on conditions (8) and (19), providing the key lemmas (Horizontal Heredity, Downward Heredity, Upward Heredity, Symmetry Lemma) with varying degrees of proof detail and leaving the remaining conditions to the reader. The section also addresses the incompleteness of Tarskian semantics for first-order relevant logics, including Fine's theorem that Tarskian semantics for RQ validates unprovable formulas. Logan formulates three explicit open problems about the extent and explanation of this incompleteness, all within reach of the Element's readership.

The eighth and final section is a deliberately biased partial survey. The most substantial portion contrasts Logan's preferred binary-operational semantics with ternary relational semantics (Routley-Meyer), identifying the core difference as the treatment of disjunction and offering a pointed critique: application of a prime to a prime need not be prime, and for strong relevant logics the logic itself fails to be prime, requiring a class of "normal points" in place of a single base point. Collection frame semantics (Restall-Standefer 2022) receives an appreciative treatment as a clever solution to associativity conditions, though Logan notes it leaves other problems unaddressed. Brief treatments follow of content semantics (Brady), algebraic semantics (Dunn), alternative proof systems (Fitch, bunched/sequent), modal and justification extensions, and further weakening below B. The coverage is useful as a guide to the literature even if, as Logan acknowledges, it cannot be comprehensive. One might note, however, that the algebraic tradition deserves more space than it receives, given its role in connecting relevance logic to the broader landscape of substructural logics.

The most distinctive contribution of the *Element* lies in its systematic recovery of the binary operational semantics of Urquhart (1972), Fine (1974), and Slaney (1990) – a tradition that, as Bimbó, Dunn, and Ferenz (2018) have documented, coexisted with the ternary relational approach from the very beginning but was historically eclipsed by it. Logan's case for the operational alternative is compelling: it avoids the structural difficulties catalogued in Section 8 and offers, through the theory-building reading inspired by Beall (2017, 2018), a philosophical transparency that the ternary relation has notoriously lacked since Copeland's critiques (1979, 1980). Recent work by Jago (2020) and Standefer (2024) confirms that this current is gaining independent momentum. One may nonetheless ask whether the presentation does not understate the resources of the ternary tradition – particularly its algebraic ramifications and its proven adaptability to a wider range of logics – and whether a less adversarial framing might better serve an introductory text.

In sum, Logan's *Relevance Logic* is not only a lucid introduction to a difficult area but also an intervention in an ongoing semantic debate. By recovering a minority tradition and situating it within a compelling philosophical narrative, the book offers both newcomers and specialists a perspective that standard references, largely organized around ternary relational semantics, often fail to capture. As the open problems presented in Section 7 make clear, the author does not gloss over foundational difficulties, but instead examines them in depth, reflecting a broader commitment to opening new lines of research. Another open question emerging from the book is whether the operational approach will once again assume a more prominent role in the field of relevance logic. What is beyond question is that Logan's *Element* makes a strong case that it deserves a place at the table.

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