Steve Nadis, Shing-Tung Yau

The Gravity of Math: How Geometry Rules the Universe

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Jacopo Rosino Giraldo
Università degli Studi di Padova

The Gravity of Math is both a historical and conceptual milestone in understanding the role of mathematics, particularly geometry, in grasping, developing, and even anticipating many crucial aspects of Einstein's Relativity. Comprising eight chapters, the book not only introduces readers to the fundamental concepts necessary to understand the core features of Relativity but goes further, retracing many of the key moments in the history of Einsteinian Relativity. Nonetheless, the book adopts an advanced mathematical approach, which may not be easily accessible to readers entirely unfamiliar with differential geometry.

Chapter 1 outlines some key moments in the history of physics, such as Newton's and Einstein's transformations of our understanding of gravity. Newton's development of calculus led to his laws of motion and the formula for gravitational attraction in the *Principia Mathematica*. However, while his law accurately described gravitational forces, it lacked an explanation of their mechanism and assumed instantaneous transmission.

Einstein's breakthrough stemmed from rejecting simultaneity, a notion Newton had taken for granted. Remarkably, even before Einstein, Mercury's anomalous precession had exposed flaws in Newton's theory. Einstein replaced absolute space and time with a four-dimensional, non-Euclidean geometry, as Minkowski had proposed. Initially skeptical, he later embraced this framework fully. The real shift in understanding gravity came with General Relativity (GR), according to which, roughly, gravity is not a force but a consequence of spacetime curvature.

Chapter 2 explores key developments in geometry by Gauss and Riemann, focusing on their contributions to the study of non-Euclidean spaces. Riemann introduced the concept of a manifold and defined distances via the metric tensor, from which the curvature tensor can be derived. A crucial property of the metric tensor is general covariance, ensuring that a manifold's intrinsic properties remain invariant under coordinate transformations.

Einstein realized that GR required a non-Euclidean geometry, but Riemannian geometry alone was insufficient. He needed a structure that reduced locally to Minkowski space, leading him to curved Lorentzian manifolds. To develop his theory, he relied on Ricci and Levi-Civita's methods for differentiation in curved spaces, using tensors to ensure coordinate-independent formulations. The chapter concludes with Einstein's temporary retreat from full general covariance due to concerns about recovering the Newtonian limit, energy-momentum conservation, and potential conflicts with the notion of causality.

Chapter 3 examines the derivation of Einstein's field equations, formulated in 1915 alongside Hilbert's independent approach. Unlike Newton and Leibniz's calculus dispute, Hilbert never claimed priority over Einstein's ideas, taking instead a more mathematical approach to physics. In *The Foundations of Physics,* Hilbert derived the equations using the principle of least action, minimizing the scalar curvature tensor, reflecting his contributions to the theory of invariants.

Emmy Noether played a key role in demonstrating energy conservation within Hilbert's framework. Her second theorem showed that in GR, energy conservation holds globally but not locally, unlike in electromagnetism. Because gravitational energy depends on the observer's position and is continuously exchanged with matter, a universally defined energy value does not exist. Conservation is only strict when considering the total energy—matter and gravity—within an isolated system from a distant perspective.

Chapter 4 explores the Einstein equations, which are ten nonlinear equations that must be solved simultaneously. Since energy, mass, and spacetime curvature are interdependent, finding exact solutions is highly complex. Even with precise initial conditions, both spacetime curvature and matter evolution must be determined together.

Karl Schwarzschild provided an exact solution for Einstein equations involving spherical masses, showing that gravity follows Newton's laws at large distances, but reveals relativistic effects near massive objects. He identified the Schwarzschild radius, beyond which nothing escapes, thus anticipating the concept of black holes. Later, Oppenheimer and Snyder demonstrated that black holes can form through gravitational collapse, while Kerr extended Schwarzschild's solution to rotating bodies.

Roger Penrose proved that singularities form regardless of symmetry, bolstering black hole theory. Schoen and Yau later confirmed that a trapped surface emerges when matter density is high enough, reinforcing black hole formation. Penrose also proposed the cosmic censorship conjecture, suggesting singularities remain hidden within event horizons, though some versions were later challenged. The chapter concludes with advanced theoretical issues on black holes, for which I suggest that the interested reader consult the text for further details.

In Chapter 5, the authors highlight Einstein's pioneering work on gravitational waves. Initially, he speculated that accelerating masses could generate them, like electromagnetic waves, but later doubted this due to the absence of negative mass. In 1916 he dismissed their existence in a letter to Schwarzschild, but soon reversed his view, formally predicting them in a 1918 paper that corrected earlier errors. Despite this, he believed they were too weak to detect.

A major breakthrough came in the mid-20th century when Yvonne Choquet-Bruhat proved that Einstein's equations could produce gravitational waves traveling at finite speeds. She also showed these equations were well-posed, ensuring stable and predictable solutions. Her work, building on Jean Leray's results, reinforced GR's mathematical foundation.

In 1991, Demetrios Christodoulou introduced the nonlinear gravitational memory effect, showing that gravitational waves leave a lasting imprint on spacetime. Later studies confirmed that other energy sources, like electromagnetic radiation, could enhance this effect. Due to the complexity of GR, formal proofs remain difficult, but Numerical Relativity has helped verify predictions and improve our understanding of gravitational waves, even if does not provide full mathematical rigor.

In Chapter 6, the authors describe how Einstein extended GR to cosmology in 1917, seeking to place it on a scientific foundation. His equations, like Newton's, faced a key issue: if gravity attracts all matter, why doesn't the universe collapse? To counter this, he introduced the cosmological constant, adding a repulsive force to maintain a static universe, the prevailing view at the time.

However, alternative models soon emerged. Willem de Sitter showed Einstein's equations allowed for an empty, expanding universe, later confirmed by Hermann Weyl and Arthur Eddington. Alexander Friedmann further demonstrated that the field equations permitted dynamic solutions, laying the groundwork for modern cosmology. Georges Lemaître built on this, proposing that the universe originated from a dense state—an idea that evolved into the Big Bang theory.

Empirical support arrived in 1964 with the discovery of the cosmic microwave background radiation, predicted by Ralph Alpher and Robert Herman. By the late 20th century, observations showed the universe's expansion was accelerating, leading to a renewed interest in the cosmological constant as a form of dark energy. Work by Roger Penrose and Stephen Hawking in the 1970s linked GR to the origins of the universe, suggesting the Big Bang was a singularity, a view later refined by quantum gravity theories. Despite its limitations, GR remains the dominant theory of gravity, though its reconciliation with quantum mechanics remains an open challenge.

In Chapter 7, the authors present the positive mass theorem. The latter states that the total mass of a spacetime satisfying certain conditions is nonnegative and zero only for Minkowski space. While GR suggests that mass and energy should be nonnegative, a rigorous proof remained elusive for decades, partly due to the challenge of defining mass in a curved spacetime.

In 1979, Schoen and Yau proved the theorem using geometric techniques based on minimal surfaces, initially for time-symmetric cases and later for general settings. In 1981, Witten provided an alternative proof using spinors and a positive energy argument, making the result more accessible to physicists. The theorem has deep implications, including connections to the Yamabe problem and the Penrose inequality, but it does not ensure the long-term stability of spacetime.

Another challenge in GR is defining mass in finite regions, known as quasilocal mass. The ADM mass, meaning the total mass-energy content of an asymptotically flat spacetime, is well-defined at spatial infinity but does not naturally extend to bounded domains. Several approaches have been proposed: Hawking's 1968 definition based on surface area, Bartnik's 1989 precise but computationally difficult formulation, and the Brown–York method (1990s), which links quasilocal mass to surface geometry but has inconsistencies in Minkowski space. More recently, Wang and Yau developed a more satisfactory definition, though it relies on solving complex partial differential equations. Despite these advances, defining mass in GR remains an open problem. The positive mass theorem rules out arbitrarily negative energy, but fundamental issues related to quasilocal mass and energy conservation continue to be active areas of research.

Chapter 8 explores the search for a unified theory of physics, particularly the challenge of merging quantum mechanics with GR into quantum gravity. While GR describes spacetime and gravity successfully, it breaks down in extreme conditions like black holes and the Big Bang. Physicists seek a broader theory that preserves its strengths while addressing its limitations.

Einstein pursued unification for decades, aiming to merge electromagnetism and gravity into a single framework. Though unsuccessful, his efforts influenced later research. Early attempts included Hermann Weyl's 1918 proposal to extend GR's equations to incorporate electromagnetism, introducing gauge invariance, now a cornerstone of modern physics. However, Einstein criticized its inconsistencies with experimental data.

In 1919, Theodor Kaluza suggested adding a fifth dimension to integrate electromagnetism with gravity, an idea refined in 1926 by Oskar Klein, who proposed that the extra dimension was compactified. While Kaluza-Klein theory ultimately failed under scrutiny, it inspired modern approaches, particularly string theory, which uses extra dimensions to unify fundamental forces. The historical trajectory from Weyl and Kaluza to contemporary gauge theories reflects the ongoing quest for unification, with Einstein's vision laying crucial groundwork for future advances.

The Gravity of Math is an extraordinary work of historical and conceptual synthesis on Einsteinian Relativity, demonstrating remarkable depth and clarity. The authors have crafted a compelling narrative that illuminates the intricate relationship between mathematical structures and physical theories. However, while their treatment of the subject is masterful, the book does not engage with the rich philosophical discussions that the principles and historical developments of Relativity have inspired. Given the scope and ambition of this work, integrating both aspects while maintaining its level of detail would have been a formidable challenge. Nonetheless, I hope that the authors might, in a future work, bring their rigorous approach to the philosophical literature as well. This could offer a much-needed bridge between historical, conceptual, and philosophical perspectives, enriching the philosophical debate on Relativity, which too often remains confined to speculation without a theoretically precise and up-to-date foundation.

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